Bull. Nov. Comp. Center, Math. Model. in Geoph., 21 (2019), 35–42

The solution of one overdetermined stationary system arising in an incompressible two-fluid medium in a half-space*

Kh.Kh. Imomnazarov, Sh.Kh. Imomnazarov, M.V. Urev, R.Kh. Bakhramov

Abstract. We have considered the classical solution in the half-space of the second boundary value problem for an overdetermined stationary system of second order equations arising in a two-fluid medium with phase equilibrium in pressure. The solution is constructed using the Fourier transform apparatus. The effect of the kinetic parameters of the medium on the solution of the system has been shown.

Keywords: Two-fluid media, incompressible fluid, Poisson equation, inhomogeneous problem, Fourier transform, classical solution.

1. The second boundary value problem for the two-velocity Stokes system with one pressure in a half-space

In the domain $\Omega = \mathbb{R}^3_+ = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, we consider the second boundary value problem for the two-velocity Stokes system with one pressure [1, 2]:

$$-\nu \boldsymbol{\Delta v} + \operatorname{grad} p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \tag{1}$$

$$-\nu_1 \boldsymbol{\Delta u} + \operatorname{grad} p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$T(\boldsymbol{v}, p)\boldsymbol{n}|_{x_3=0} = \boldsymbol{a}(x_1, x_2), \quad T(\boldsymbol{u}, p)\boldsymbol{n}|_{x_3=0} = \boldsymbol{b}(x_1, x_2), \tag{3}$$

where $\boldsymbol{n} = (0, 0, -1)$ denotes the normal to $\partial\Omega$ at $x_3 = 0$, ν, ν_1 are the kinematic viscosity coefficients, $\boldsymbol{a}, \boldsymbol{b}$ are given functions, $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{u} = (u_1, u_2, u_3)$ are unknown vector fields of velocities, p is pressure, $T(\boldsymbol{v}, p)$, $T(\boldsymbol{u}, p)$ are the stress tensors corresponding to the flows $\boldsymbol{v}, p$ and $\boldsymbol{u}, p$:

$$T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v}), \quad S(\boldsymbol{v}) = \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)_{i,j=1,2,3},$$

$$T(\boldsymbol{u}, p) = -pI + \nu_1 S(\boldsymbol{u}), \quad S(\boldsymbol{u}) = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)_{i,j=1,2,3}.$$

Here I is the unit matrix, $S(\boldsymbol{v}), S(\boldsymbol{u})$ are the doubled strain rate tensors.

*Supported by the Russian Foundation for Basic Research under Grant 18-51-41002.

2. The half-space problem for a homogeneous system

Consider the homogeneous problem for system (1)–(3) in $\mathbb{R}^3_+$:

$$\begin{cases} -\nu\mathbf{\Delta}\boldsymbol{v} + \operatorname{grad} p = \mathbf{0}, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3_+, \\ \nu\Big(\dfrac{\partial v_j}{\partial x_3} + \dfrac{\partial v_3}{\partial x_j}\Big)\Big|_{x_3=0} = -a_j(x_1,x_2), \quad j = 1,2, \\ \Big(-p + 2\nu\dfrac{\partial v_3}{\partial x_3}\Big)\Big|_{x_3=0} = -a_3(x_1,x_2); \end{cases} \tag{4}$$

$$\begin{cases} -\nu_1\mathbf{\Delta}\boldsymbol{u} + \operatorname{grad} p = \mathbf{0}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3_+, \\ \nu_1\Big(\dfrac{\partial u_j}{\partial x_3} + \dfrac{\partial u_3}{\partial x_j}\Big)\Big|_{x_3=0} = -b_j(x_1,x_2), \quad j = 1,2, \\ \Big(-p + 2\nu_1\dfrac{\partial u_3}{\partial x_3}\Big)\Big|_{x_3=0} = -b_3(x_1,x_2), \end{cases} \tag{5}$$

assuming that the functions $a_m(x_1,x_2)$ and $b_m(x_1,x_2)$ $(m = 1,2,3)$ are smooth and decrease sufficiently rapidly for $|(x_1,x_2)| \to \infty$. The solution should also decrease at infinity.

The solution of system (4), (5) is reduced to the sequential solution of two boundary value problems. First, the Stokes problem (4) is solved for $(\boldsymbol{v},p)$ [3–5], and then the second velocity $\boldsymbol{u}$ is determined as a solenoidal solution to the following boundary value problem for the Poisson vector equation:

$$\begin{cases} \nu_1\mathbf{\Delta}\boldsymbol{u} = \operatorname{grad} p, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3_+, \\ \nu_1\Big(\dfrac{\partial u_j}{\partial x_3} + \dfrac{\partial u_3}{\partial x_j}\Big)\Big|_{x_3=0} = -b_j(x_1,x_2), \quad j = 1,2, \\ \dfrac{\partial u_3}{\partial x_3}\Big|_{x_3=0} = \dfrac{1}{2\nu_1}(p(x_1,x_2,0) - b_3(x_1,x_2)). \end{cases} \tag{6}$$

Denote by $\tilde{g}(\alpha_1,\alpha_2,x_3)$ Fourier transform of the function $g(x_1,x_2,x_3)$ with respect to the variables x_1,x_2 [3–5]:

$$\tilde{g}(\alpha_1,\alpha_2,x_3) = \frac{1}{(2\pi)^{3/2}}\int_{R^2} e^{-i\alpha_1x_1 - i\alpha_2x_2} g(x_1,x_2,x_3)\,dx_1\,dx_2.$$

After applying the Fourier transform with respect to the variables x_1,x_2 to system (4), for the transformed functions $\tilde{v}_k,\tilde{p}$, we obtain the boundary value problem for a system of ordinary differential equations on the semi-axis $(0,\infty)$ with the parameter $\alpha = (\alpha_1,\alpha_2)$:

$$\begin{cases} \nu|\alpha|^2\tilde{v}_j - \nu\dfrac{d^2\tilde{v}_j}{dx_3^2} + i\alpha_j\tilde{p} = 0, \quad j = 1,2, \\ \nu|\alpha|^2\tilde{v}_3 - \nu\dfrac{d^2\tilde{v}_3}{dx_3^2} + \dfrac{d\tilde{p}}{dx_3} = 0, \\ i\alpha_1\tilde{v}_1 + i\alpha_2\tilde{v}_2 + \dfrac{d\tilde{v}_3}{dx_3} = 0, \end{cases} \tag{7}$$

$$\begin{cases} \nu\Big(\dfrac{d\tilde{v}_j}{dx_3} + i\alpha_j\tilde{v}_3\Big)\Big|_{x_3=0} = -\tilde{a}_j, \quad j = 1,2, \\ \Big(-\tilde{p} + 2\nu\dfrac{d\tilde{v}_3}{dx_3}\Big)\Big|_{x_3=0} = -\tilde{a}_3, \\ \tilde{\boldsymbol{v}} \to \mathbf{0} \ \text{ at } \ x_3 \to \infty. \end{cases} \tag{8}$$

The general solution of the system of equations (7) tending to zero when $x_3 \to \infty$ has the following form:

$$\begin{aligned} \tilde{v}_1(\alpha, x_3) &= \Big(\frac{\alpha_2}{|\alpha|^2}C_1 - \frac{i\alpha_1}{|\alpha|}C_2 + \frac{i\alpha_1}{2\nu|\alpha|}\Big(\frac{1}{|\alpha|} - x_3\Big)C_3\Big)e^{-|\alpha|x_3}, \\ \tilde{v}_2(\alpha, x_3) &= \Big(-\frac{\alpha_1}{|\alpha|^2}C_1 - \frac{i\alpha_2}{|\alpha|}C_2 + \frac{i\alpha_2}{2\nu|\alpha|}\Big(\frac{1}{|\alpha|} - x_3\Big)C_3\Big)e^{-|\alpha|x_3}, \\ \tilde{v}_3(\alpha, x_3) &= \Big(C_2 + \frac{1}{2\nu}x_3C_3\Big)e^{-|\alpha|x_3}, \\ \tilde{p}(\alpha, x_3) &= C_3e^{-|\alpha|x_3}. \end{aligned} \tag{9}$$

The constants C_1, C_2, C_3 are determined from boundary conditions (8). The solution to problem (7), (8) can be represented as follows:

$$\tilde{\boldsymbol{v}} = -\tilde{a}_1\tilde{\boldsymbol{v}}^1 - \tilde{a}_2\tilde{\boldsymbol{v}}^2 - \tilde{a}_3\tilde{\boldsymbol{v}}^3, \quad \tilde{p} = -\tilde{a}_1\tilde{p}^1 - \tilde{a}_2\tilde{p}^2 - \tilde{a}_3\tilde{p}^3, \tag{10}$$

where $\tilde{\boldsymbol{v}}^1(\alpha, x_3)$, $\tilde{p}^1(\alpha, x_3)$ is the solution (9) of system (7) with the constants

$$C_1 = -\frac{\alpha_2}{\nu|\alpha|}, \quad C_2 = 0, \quad C_3 = \frac{i\alpha_1}{|\alpha|},$$

satisfying boundary conditions (8) with the vector $\boldsymbol{e}_1 = (1,0,0)$ in the right-hand side:

$$\begin{aligned} \tilde{v}_1^1(\alpha, x_3) &= \Big(-\frac{1}{2\nu|\alpha|} - \frac{\alpha_2^2}{2\nu|\alpha|^3} + \frac{\alpha_1^2}{2\nu|\alpha|^2}x_3\Big)e^{-|\alpha|x_3}, \\ \tilde{v}_2^1(\alpha, x_3) &= \Big(\frac{\alpha_1\alpha_2}{2\nu|\alpha|^3} + \frac{\alpha_1\alpha_2}{2\nu|\alpha|^2}x_3\Big)e^{-|\alpha|x_3}, \\ \tilde{v}_3^1(\alpha, x_3) &= \frac{i\alpha_1}{2\nu|\alpha|}x_3\, e^{-|\alpha|x_3}, \\ \tilde{p}^1(\alpha, x_3) &= \frac{i\alpha_1}{|\alpha|}e^{-|\alpha|x_3}; \end{aligned} \tag{11}$$

$\tilde{\boldsymbol{v}}^2(\alpha, x_3)$, $\tilde{p}^2(\alpha, x_3)$ is the solution (9) of system (7) with the constants

$$C_1 = \frac{\alpha_1}{\nu|\alpha|}, \quad C_2 = 0, \quad C_3 = \frac{i\alpha_2}{|\alpha|},$$

satisfying boundary conditions (8) with the vector $\boldsymbol{e}_2 = (0, 1, 0)$ in the right-hand side:

$$\begin{aligned} \tilde{v}_1^2(\alpha, x_3) &= \Big(\frac{\alpha_1\alpha_2}{2\nu|\alpha|^3} + \frac{\alpha_1\alpha_2}{2\nu|\alpha|^2} x_3\Big) e^{-|\alpha|x_3}, \\ \tilde{v}_2^2(\alpha, x_3) &= \Big(-\frac{1}{2\nu|\alpha|} - \frac{\alpha_1^2}{2\nu|\alpha|^3} + \frac{\alpha_2^2}{2\nu|\alpha|^2} x_3\Big) e^{-|\alpha|x_3}, \\ \tilde{v}_3^2(\alpha, x_3) &= \frac{i\alpha_2}{2\nu|\alpha|} x_3\, e^{-|\alpha|x_3}, \\ \tilde{p}^2(\alpha, x_3) &= \frac{i\alpha_2}{|\alpha|} e^{-|\alpha|x_3}; \end{aligned} \tag{12}$$

and $\tilde{\boldsymbol{v}}^3(\alpha, x_3)$, $\tilde{p}^3(\alpha, x_3)$ is the solution (9) of system (7) with the constants

$$C_1 = 0, \quad C_2 = -\frac{1}{2\nu|\alpha|}, \quad C_3 = -1,$$

satisfying boundary conditions (8) with the vector $\boldsymbol{e}_3 = (0, 0, 1)$ in the right-hand side:

$$\begin{aligned} \tilde{v}_1^3(\alpha, x_3) &= \frac{i\alpha_1}{2\nu|\alpha|} x_3 e^{-|\alpha|x_3}, \\ \tilde{v}_2^3(\alpha, x_3) &= \frac{i\alpha_2}{2\nu|\alpha|} x_3 e^{-|\alpha|x_3}, \\ \tilde{v}_3^3(\alpha, x_3) &= -\frac{1}{2\nu}\Big(\frac{1}{|\alpha|} + x_3\Big)\, e^{-|\alpha|x_3}, \\ \tilde{p}^3(\alpha, x_3) &= -e^{-|\alpha|x_3}. \end{aligned} \tag{13}$$

Denote $|\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. When performing the inverse Fourier transform in formulas (11)–(13), we use the equations

$$\begin{aligned} F^{-1}\Big[\frac{1}{|\alpha|} e^{-|\alpha|x_3}\Big] &= \frac{1}{2\pi}\int_{R^2} \frac{1}{|\alpha|} e^{ix_1\alpha_1 + ix_2\alpha_2 - |\alpha|x_3} d\alpha_1 d\alpha_2 = \frac{1}{|\boldsymbol{x}|}, \\ F^{-1}\Big[e^{-|\alpha|x_3}\Big] &= \frac{x_3}{|\boldsymbol{x}|^3}, \\ F^{-1}\Big[\frac{\alpha_1}{|\alpha|} e^{-|\alpha|x_3}\Big] &= \frac{1}{2\pi}\int_{R^2} \frac{\alpha_1}{|\alpha|} e^{ix_1\alpha_1 + ix_2\alpha_2 - |\alpha|x_3} d\alpha_1 d\alpha_2 = \frac{ix_1}{|\boldsymbol{x}|^3}, \\ F^{-1}\Big[\frac{\alpha_1\alpha_2}{|\alpha|^2} e^{-|\alpha|x_3}\Big] &= \frac{1}{2\pi}\int_{R^2} \frac{\alpha_1\alpha_2}{|\alpha|^2} e^{ix_1\alpha_1 + ix_2\alpha_2 - |\alpha|x_3} d\alpha_1 d\alpha_2 \\ &= 3x_1x_2 \int \frac{dx_3}{|\boldsymbol{x}|^5} = \frac{x_1x_2x_3(3x_1^2 + 3x_2^2 + 2x_3^2)}{(x_1^2 + x_2^2)^2|\boldsymbol{x}|^3}, \end{aligned}$$

$$F^{-1}\Big[\frac{\alpha_1\alpha_2}{|\alpha|^3}e^{-|\alpha|x_3}\Big] = \frac{1}{2\pi}\int_{R^2}\frac{\alpha_1\alpha_2}{|\alpha|^3}e^{ix_1\alpha_1+ix_2\alpha_2-|\alpha|x_3}d\alpha_1 d\alpha_2$$
$$= -\frac{x_1x_2(x_1^2+x_2^2+2x_3^2)}{(x_1^2+x_2^2)^2|\boldsymbol{x}|},$$
$$F^{-1}\Big[\frac{\alpha_1^2}{|\alpha|^2}e^{-|\alpha|x_3}\Big] = \frac{x_3}{(x_1^2+x_2^2)|\boldsymbol{x}|}\Big(\frac{2x_1^2}{x_1^2+x_2^2}+\frac{x_1^2}{|\boldsymbol{x}|^2}-1\Big),$$
$$F^{-1}\Big[\frac{\alpha_2^2}{|\alpha|^3}e^{-|\alpha|x_3}\Big] = \frac{x_2^2}{(x_1^2+x_2^2)|\boldsymbol{x}|} - \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}|\boldsymbol{x}|.$$

Performing the inverse Fourier transform in (11) using the above formulas, we obtain

$$v_1^1(\boldsymbol{x}) = -\frac{x_1^2+2x_2^2}{2\nu(x_1^2+x_2^2)|\boldsymbol{x}|} + \frac{(x_2^2-x_1^2)|\boldsymbol{x}|}{2\nu(x_1^2+x_2^2)^2} +$$
$$\frac{x_3^2}{2\nu(x_1^2+x_2^2)|\boldsymbol{x}|}\Big(\frac{2x_1^2}{x_1^2+x_2^2}+\frac{x_1^2}{|\boldsymbol{x}|^2}-1\Big),$$
$$v_2^1(\boldsymbol{x}) = -\frac{x_1x_2(x_1^2+x_2^2+2x_3^2)}{2\nu(x_1^2+x_2^2)^2|\boldsymbol{x}|} + \frac{x_1x_2x_3^2(3x_1^2+3x_2^2+2x_3^2)}{2\nu(x_1^2+x_2^2)^2|\boldsymbol{x}|^3},$$
$$v_3^1(\boldsymbol{x}) = \frac{-x_1x_3}{2\nu|\boldsymbol{x}|^3}, \qquad p^1(\boldsymbol{x}) = \frac{-x_1}{|\boldsymbol{x}|^3}.$$

Similarly, from (12) and (13), we have

$$v_1^2(\boldsymbol{x}) = -\frac{x_1x_2}{2\nu(x_1^2+x_2^2)^2|\boldsymbol{x}|}\Big(x_1^2+x_2^2+2x_3^2-\frac{x_3^2(3x_1^2+3x_2^2+2x_3^2)}{|\boldsymbol{x}|^2}\Big),$$
$$v_2^2(\boldsymbol{x}) = -\frac{2x_1^2+x_2^2}{2\nu(x_1^2+x_2^2)|\boldsymbol{x}|} + \frac{(x_1^2-x_2^2)|\boldsymbol{x}|}{2\nu(x_1^2+x_2^2)^2} +$$
$$\frac{x_3^2}{2\nu(x_1^2+x_2^2)|\boldsymbol{x}|}\Big(\frac{2x_2^2}{x_1^2+x_2^2}+\frac{x_2^2}{|\boldsymbol{x}|^2}-1\Big),$$
$$v_3^2(\boldsymbol{x}) = \frac{-x_2x_3}{2\nu|\boldsymbol{x}|^3}, \qquad p^2(\boldsymbol{x}) = \frac{-x_2}{|\boldsymbol{x}|^3}.$$
$$v_1^3(\boldsymbol{x}) = \frac{-x_1x_3}{2\nu|\boldsymbol{x}|^3}, \qquad v_2^3(\boldsymbol{x}) = \frac{-x_2x_3}{2\nu|\boldsymbol{x}|^3},$$
$$v_3^3(\boldsymbol{x}) = -\frac{x_1^2+x_2^2+2x_3^2}{2\nu|\boldsymbol{x}|^3}, \qquad p^3(\boldsymbol{x}) = -\frac{x_3}{|\boldsymbol{x}|^3}.$$

To perform the inverse Fourier transform in equations (10), let us apply the well-known formula

$$F[(f*g)] = 2\pi F[f]F[g] = 2\pi\tilde{f}\tilde{g},$$

from where

$$F^{-1}[\tilde{f}\tilde{g}] = \frac{1}{2\pi}\big(F^{-1}[\tilde{f}]*F^{-1}[\tilde{g}]\big) = \frac{1}{2\pi}(f*g).$$

Here

$$(f * g)(\boldsymbol{x}) = \int_{R^2} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y}$$

is the convolution of the functions f and g.

Then, from (10), we have

$$\boldsymbol{v} = -\frac{1}{2\pi}[(a_1 * \boldsymbol{v}^1) - (a_2 * \boldsymbol{v}^2) - (a_3 * \boldsymbol{v}^3)],$$
$$p = -\frac{1}{2\pi}[(a_1 * p^1) - (a_2 * p^2) - (a_3 * p^3)].$$

Let us turn to the solution of problem (6). The solution $\boldsymbol{u}$ will be sought for in the form $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{w}$. We extend the solenoidal function ∇p with $\mathbb{R}^3_+$ to the whole space $\mathbb{R}^3$ while maintaining solenoidality and smoothness. Such a continuation is possible (see, for example, [3, Lemma 4]). Setting

$$\boldsymbol{z}(\boldsymbol{x}) = -\frac{1}{4\pi\nu_1} \int_{\mathbb{R}^3} \frac{\nabla p(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}\, d\boldsymbol{y},$$

we obtain with confidence the solenoidal solution $\boldsymbol{z} = (z_1, z_2, z_3)$ of the Poisson vector equation

$$\nu_1 \boldsymbol{\Delta z} = \operatorname{grad} p.$$

The function $\boldsymbol{w} = (w_1, w_2, w_3)$ must be solenoidal and satisfy the Laplace vector equation (6) with modified boundary conditions:

$$\begin{cases} \boldsymbol{\Delta w} = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in } \mathbb{R}^3_+, \\ \nu_1 \Big(\dfrac{\partial w_j}{\partial x_3} + \dfrac{\partial w_3}{\partial x_j} \Big)\Big|_{x_3=0} = -d_j(x_1, x_2), \quad j = 1, 2, \\ \Big(-p + 2\nu_1 \dfrac{\partial w_3}{\partial x_3} \Big)\Big|_{x_3=0} = -d_3(x_1, x_2), \end{cases} \tag{14}$$

where

$$d_j(x_1, x_2) = b_j(x_1, x_2) + \nu_1 \Big(\frac{\partial z_j}{\partial x_3} + \frac{\partial z_3}{\partial x_j} \Big)\Big|_{x_3=0}, \quad j = 1, 2,$$
$$d_3(x_1, x_2) = b_3(x_1, x_2) + 2\nu_1 \frac{\partial z_3}{\partial x_3}\Big|_{x_3=0}.$$

After applying the Fourier transform with respect to the variables x_1, x_2 to system (14), for the transformed functions we obtain $\tilde{w}_k$ boundary value problem for the overdetermined system of ordinary differential equations on the semi-axis $(0, \infty)$ with the parameter $\alpha = (\alpha_1, \alpha_2)$:

$$\begin{cases} \dfrac{d^2\tilde{\boldsymbol{w}}}{dx_3^2} - |\alpha|^2\tilde{\boldsymbol{w}} = \boldsymbol{0}, \\ i\alpha_1\tilde{w}_1 + i\alpha_2\tilde{w}_2 + \dfrac{d\tilde{w}_3}{dx_3} = 0, \end{cases} \tag{15}$$

$$\begin{cases} \Big(\dfrac{d\tilde{w}_j}{dx_3} + i\alpha_j\tilde{w}_3\Big)\Big|_{x_3=0} = -\dfrac{\tilde{d}_j}{\nu_1}, \quad j = 1, 2, \\ -\tilde{p} + 2\nu_1\dfrac{d\tilde{w}_3}{dx_3}\Big|_{x_3=0} = -\tilde{d}_3, \\ \tilde{\boldsymbol{w}} \to \boldsymbol{0} \text{ at } x_3 \to \infty. \end{cases} \tag{16}$$

The general solution of the first vector equation in (15), tending to zero when $x_3 \to \infty$, has the following form:

$$\tilde{\boldsymbol{w}}(\alpha, x_3) = \boldsymbol{C}e^{-|\alpha|x_3}, \quad \boldsymbol{C} = (C_1, C_2, C_3). \tag{17}$$

The constants C_1, C_2, C_3 are determined from boundary conditions (16):

$$C_j = -\frac{i\alpha_j}{2|\alpha|^2\nu_1}\big(\tilde{p}|_{x_3=0} - \tilde{d}_3\big) + \frac{\tilde{d}_j}{|\alpha|\nu_1}, \quad j = 1, 2,$$

$$C_3 = -\frac{1}{2|\alpha|\nu_1}\big(\tilde{p}|_{x_3=0} - \tilde{d}_3\big).$$

In addition, there must be a divergent condition on the solution, which gives a restriction on the functions of the right-hand side boundary conditions (16). From

$$i\alpha_1 C_1 + i\alpha_2 C_2 - |\alpha| C_3 = 0$$

follows the equality

$$\frac{i\alpha_1}{|\alpha|}\tilde{d}_1 + \frac{i\alpha_2}{|\alpha|}\tilde{d}_2 + \tilde{p}|_{x_3=0} - \tilde{d}_3 = 0. \tag{18}$$

Performing the inverse Fourier transform in (17) and (18), we obtain

$$w_1(\boldsymbol{x}) = -\frac{i}{2\nu_1}F^{-1}\Big[\frac{\alpha_1}{|\alpha|^2}e^{-|\alpha|x_3}\Big] * (p|_{x_3=0} - d_3) + \frac{1}{\nu_1}F^{-1}\Big[\frac{1}{|\alpha|}e^{-|\alpha|x_3}\Big] * d_1,$$

$$w_2(\boldsymbol{x}) = -\frac{i}{2\nu_1}F^{-1}\Big[\frac{\alpha_2}{|\alpha|^2}e^{-|\alpha|x_3}\Big] * (p|_{x_3=0} - d_3) + \frac{1}{\nu_1}F^{-1}\Big[\frac{1}{|\alpha|}e^{-|\alpha|x_3}\Big] * d_2,$$

$$w_3(\boldsymbol{x}) = -\frac{1}{2\nu_1}F^{-1}\Big[\frac{1}{|\alpha|}e^{-|\alpha|x_3}\Big] * (p|_{x_3=0} - d_3).$$

The overdetermination of problem (14) results in the necessary condition of its solvability in the form of the performance requirements

$$iF^{-1}\Big(\frac{\alpha_1}{|\alpha|}\Big) * d_1 + iF^{-1}\Big(\frac{\alpha_2}{|\alpha|}\Big) * d_2 + p|_{x_3=0} - d_3 = 0.$$

for the components of the vector $\boldsymbol{d}$.

References

[1] Zhuraev D., Tan Zhian-Gan, Imomnazarov Kh.Kh., Urev M.V. The boundary problem for one overdetermined system, arising in a two-fluid environment // UzbMZh. — 2016. — No. 3. — P. 58–69 (In Russian).

[2] Urev M.V., Imomnazarov Kh.Kh., Zhian-Gan Tan. A boundary value problem for one overdetermined stationary system arising in a two-velocity hydrodynamics // Sib. J. Num. Math. — 2017. — Vol. 20, No. 4. — P. 425–437 (In Russian).

[3] Solonnikov V.A. Estimates for nonstationary linearized solutions Navier–Stokes equations // Trudy MIAN USSR. — 1964. — Vol. 70. — P. 213–317 (In Russian).

[4] Solonnikov V.A. Estimates for the solution of one initial-boundary value problem for a linear non-stationary Navier–Stokes equations systems // Zapiski. Scientific Seminar. LOMI. — 1976. — Vol. 59. — P. 178–254 (In Russian).

[5] Solonnikov V.A. Estimates for the solution of the second initial-boundary value problem for the Stokes system in the spaces functions with derivatives continuous in Holder with respect to spatial variables // Zapiski. Scientific Seminar. POMI. — 1999. — Vol. 259. — P. 254–279 (In Russian).

[6] Bondar L.N. Solvability of the second boundary value problem for the Stokes system // Sib. J. Industry Math. — 2014. — Vol. 17, No. 3. — P. 26–39 (In Russian).