Bull. Nov. Comp. Center, Comp. Science, 43 (2019), 21–38

Transition system semantics for flow event structures*

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Abstract. In this paper, we deal with event-oriented models of concurrent processes which are generalizations of the well-studied model of prime event structures. In particular, we translate flow event structures into structures for resolvable conflict (the most expressive event-oriented model) and back, define two structurally different methods of generating transition systems from the models under consideration, and demonstrate that, despite their differences, the methods lead to isomorphic results in interleaving and step semantics.

Keywords: Flow event structures, interleaving and step semantics, behaviour, transition systems

1. Introduction

Flow event structures [5] extend the well-studied class of prime event structures [22] in several ways: in flow structures, the causality ordering represented by a flow relation is no longer a partial order; the symmetric conflict relation can be reflexive; the flow and the conflict relations can overlap; the principles of finite causes and conflict inheritance are dropped.

Transition systems play an important role in concurrency theory. Associating a transition system with a true concurrency model has proved to be a suitable technique for studying various problems related to reactive systems including consistency, bisimulation, implementation and verification. Two structurally different methods of associating transition system semantics to event structure models are distinguished in the literature. One of them is based on configurations (states are sets of executed events), e.g., see [1, 2, 8, 9, 10, 11, 13, 14, 19, 21]), the other on residuals (states are model fragments left after a partial execution of the model), e.g., see [5, 6, 12, 14, 15, 16, 17, 20]. Configuration-based transition systems seem to be predominantly used as the semantics of event structures, but residual-based transition systems are actively used in providing operational semantics of process calculi and in demonstrating the consistency of operational and denotational semantics.

In the literature, the two semantics have occasionally been used alongside each other (see [14] as an example), but their general relationship has not

*Supported by German Research Foundation under Grant Be 1267/14-1

been studied too deeply. In a seminal paper, viz. [18], bisimulations between configuration-based and residual-based transition systems have been proved to exist for prime event structures [22]. This result has been extended in [3] to more complex event structure models. A crucial technical subtlety pertains to the *removal operator* that lies at the heart of residual semantics. Counterexamples illustrate that an isomorphism cannot be achieved with the various removal operators defined in [3, 18]. The paper [4] demonstrates that, nevertheless, the removal operators can be tightened in such a way that *isomorphisms*, rather than just bisimulations, between the two types of transition systems belonging to a single event structure can be obtained.

In this paper, we consider flow event structures and translate them into event structures for resolvable conflict (the most expressive event-based model) and back. Also, we define removal operators for the models under consideration and provide isomorphism results on the two kinds of transition systems belonging to a single event structure, in interleaving and step semantics.

The rest of paper is organized as follows. In Section 2, the models of flow event structures and event structures for resolvable conflict are considered, translations between them are provided, and removal operators for the models are developed. Section 3 contains the definitions of construction operators of two types of transition systems from the event structure models that are proven to lead to isomorphism results. Section 4 concludes. The proofs of the results obtained can be found in Appendix.

2. Event structure models

2.1. Flow event structures

Flow event structures introduced in [5] are another kind of event structures having a similar representation as prime event structures[1] [22] but being much more relaxed. First, the causality ordering in flow event structures is represented by an irreflexive flow relation that is not necessarily transitive and acyclic. Second, the symmetric conflict relation is not assumed to be irreflexive; this means that self-conflicting events are allowed. Such events cannot in general be removed from a flow structure without affecting its set of configurations. Third, there is no requirement on the relationships between the flow and the conflict relations. Fourth, the principles of finite causes and conflict inheritance are dropped.

[1]A prime event structure is a tuple $\mathcal{E} = (E, \sharp, \leq)$, where E is a set of events; $\leq \subseteq E \times E$ is a partial order (the *causality relation*), satisfying the *principle of finite causes*: $\forall e \in E\colon \lfloor e \rfloor = \{e' \in E \mid e' \leq e\}$ is finite; $\sharp \subseteq E \times E$ is an irreflexive and symmetric relation (the *conflict relation*), satisfying the *principle of hereditary conflict*: $\forall e, e', e'' \in E\colon e \leq e'$ and $e \sharp e''$ then $e' \sharp e''$.

Definition 1. *A* flow event structure (F-structure) over L *is a tuple* $\mathcal{E}$ $= (E,\ \sharp,\ \prec,\ L,\ l)$*, where E is a set of events; $\sharp \subseteq E \times E$ is a symmetric relation (the* conflict relation*); $\prec\ \subseteq E \times E$ is an irreflexive relation (the* flow relation*); L is a set of labels; $l : E \to L$ is a labeling function.*

Consider the notion of a configuration of F-structures. First, configurations must be finite and, moreover, conflict-free (hence, self-conflicting events will never occur in any configuration, i.e. they are impossible). Second, for an event to occur it is necessary that a complete non-conflicting set of its immediate causes has occurred. Here, we say that d is a *possible immediate cause* for e iff $d \prec e$, and a set of immediate causes is *complete* if for any cause which is not contained there is a conflicting cause which is included. Third, no cycles with respect to causal dependence may occur. A set $X \subseteq E$ is a *configuration* of an F-structure $\mathcal{E}$ iff X is a finite set, conflict-free (i.e., for all $e, e' \in X$ $\neg(e \ \sharp\ e')$), left-closed up to conflicts (i.e., for all $d, e \in E$ if $e \in X$, $d \prec e$ and $d \notin X$ then there is $f \in X$ such that $d \ \sharp\ f \prec e$), and does not contain flow cycles. The set of configurations of $\mathcal{E}$ is denoted $Conf(\mathcal{E})$.

In the graphical representation of an F-structure, $e \prec e'$ is drawn as an arrow from e to e'; the pairs of the events included in the conflict relation are marked by the symbol $\sharp$; and the self-conflicts are pictured as dotted circles around the events.

$\mathcal{E}^f$:

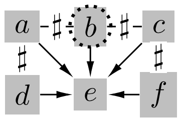

Figure 1. A flow event structure $\mathcal{E}^f$

Example 1. *Figure 1 presents the F-structure $\mathcal{E}^f$ over $L = \{a, b, c, d, e, f\}$, with $E^f = L$; $\sharp^f = \{(a, b),\ (b, a),\ (b, b),\ (b, c),\ (c, b),\ (a, d),\ (d, a),\ (c, f),$ $(f, c)\}$; $\prec^f = \{(d, e),\ (a, e),\ (b, e),\ (c, e)\ (f, e)\}$; and the identity labeling function l^f. The set of configurations $Conf(\mathcal{E}^f)$ consists of the sets: $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{f\}$, $\{a, c\}$, $\{a, f\}$, $\{c, d\}$, $\{d, f\}$, $\{a, c, e\}$, $\{a, f, e\}$, $\{c, d, e\}$.*

We are ready to define the removal operator of F-structures.

Definition 2. *For $\mathcal{E} = (E,\ \prec,\ \sharp,\ L,\ l) \in \mathbb{E}^f_L$ and $X \in Conf(\mathcal{E})$, a removal operator is defined as follows: $\mathcal{E} \setminus X = (E',\ \sharp',\ \prec',\ L,\ l')$, with*

$$\begin{array}{lll} E' & = & E \setminus X \\ \sharp' & = & (\sharp\ \cap\ (E' \times E'))\ \cup\ \{(e, e) \mid e \in \sharp(X)\}, \\ & & \qquad\qquad\qquad\qquad \textit{where } \sharp(X) = \{e' \in E \mid \exists e \in X\colon e\ \sharp\ e'\} \\ \prec' & = & (\prec\ \cap\ (E' \times E')) \setminus \{(e, f) \in \prec \mid \exists e' \in X\ :\ e\ \sharp\ e' \prec f\} \\ l' & = & l\ |_{E'}. \end{array}$$

The intuitive interpretation of the above definition is the following. All the events in X are removed from E; the conflict relation $\sharp'$ contains the pairs of remaining conflicting events and newly-added self-conflicting events being in conflict with some events in X; and the flow relation $\prec'$ includes the pairs of remaining events related by $\prec$ without the pairs (e, f) with e conflicting with some e' in X and f having immediate causes e and e'. We remove the pairs (e, f) because e and e' being in conflict belong to different complete sets of causes of f and the self-conflicting event e not being in conflict with the events from the intersection of the complete sets would prohibit a possible execution of f.

Consider properties of the removal operator.

Lemma 1. *Given an F-structure $\mathcal{E}$ and $X \in Conf(\mathcal{E})$,*

(i) $\mathcal{E} \setminus X$ *is an F-structure;*

(ii) $X \subseteq Y \in Conf(\mathcal{E}) \Rightarrow Y \setminus X \in Conf(\mathcal{E} \setminus X)$.

2.2. Event structures for resolvable conflict

In this section, we consider event structures for resolvable conflict, which were put forward in [10] to give semantics to general Petri nets. A structure for resolvable conflict consists of a set of events and an enabling relation $\vdash$ between sets of events. The enabling $X \vdash Y$ with sets X and Y imposes restrictions on the occurrences of events in Y by requiring that for all events in Y to occur, their *causes* – the events in X – have to occur before. This allows for modeling the case when a and b cannot occur together until c occurs, i.e., initially a and b are in conflict until the occurrence of c resolves this conflict. Notice that flow event structures are unable to model the phenomena of resolvable conflict. In resolvable conflict structures, the enabling relation can also model conflicts: events from a set Y are *in irresolvable conflict* iff there is no enabling of the form $X \vdash Y$ for any set X of events.

Definition 3. *An* event structure for resolvable conflict (RC-structure) over *L is a tuple $\mathcal{E} = (E, \vdash, L, l)$, where E is a set of events; $\vdash \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$ is the* enabling relation*; L is a set of labels; $l : E \to L$ is a labeling function.*

Let $\mathcal{E}$ be an RC-structure over L, $X \subseteq E$, and $e \in E$. We write $Con(X)$ iff $\forall \widehat{X} \subseteq X\colon \exists Z \subseteq E\colon Z \vdash \widehat{X}$, and $Con_i(X)$ iff $\forall \widehat{X} \subseteq X\colon \mid \widehat{X} \mid = i\colon \exists Z \subseteq E\colon Z \vdash \widehat{X}$ ($i \in \{1, 2\}$). The *direct causality relation* $\prec^{rc} \subseteq E \times E$ is defined as follows: $d \prec^{rc} e \iff \exists X \subseteq E\colon X \vdash_{min} e$, $d \in X$ and $Con_2(X)$. The *immediate conflict relation* $\sharp^{rc} \subseteq E \times E$ is given by: $d \sharp^{rc} e \iff (d = e \Rightarrow \neg Con_1(\{d\})) \vee (d \neq e \Rightarrow \neg Con_2(\{d, e\}))$. We also determine the (strong) *conflict relation* $\sharp \subseteq E \times E$ as follows: $d \sharp e \iff \neg Con(\{d, e\})$.

Let $\downarrow^{rc} e$ be a maximal subset of E such that $\forall e' \in \downarrow^{rc} e\colon e' \prec^{rc} e$ and $\forall e' \neq e'' \in \downarrow^{rc} e\colon \neg(e' \sharp^{rc} e'')$.

A set $X \subseteq E$ is *left-closed* iff X is finite, and for all $\widetilde{X} \subseteq X$ there exists a set $\widehat{X} \subseteq X$ such that $\widehat{X} \vdash \widetilde{X}$. The set of the left-closed sets of $\mathcal{E}$ is denoted as $LC(\mathcal{E})$. Clearly, any left-closed set is conflict-free. A set $X \subseteq E$ is a *configuration of* $\mathcal{E}$ iff X can be represented as an ordered set $\{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash Y$. Let $Conf(\mathcal{E})$ be the set of configurations of $\mathcal{E}$. Clearly, any configuration X is a left-closed set but not conversely.

The *direct causality relation within a configuration* $X \in Conf(\mathcal{E})$ and *within a subset* $X' \subseteq X$ are respectively defined as follows: $\prec^{rc}_X = \{(e_i, e_j) \in X \times X \mid \forall Y \subseteq X\colon (Y \vdash e_j \Rightarrow e_i \in Y)$, and $\prec^{rc}_{X'} = \{(e_i, e_j) \in X' \times X' \mid e_i \prec^{rc}_X e_j\}$. We take $\leq^{rc}_{X'}$ as the reflexive and transitive closure of $\prec^{rc}_{X'}$.

Consider some properties of event structures for resolvable conflict.

Definition 4. *An RC-structure $\mathcal{E} = (E, \vdash, L, l)$ is*

- rooted *iff $(\emptyset, \emptyset) \in \vdash$;*
- pure *iff $X \vdash Y \Rightarrow X \cap Y = \emptyset$;*
- singular *iff $X \vdash Y \Rightarrow X = \emptyset \vee \mid Y \mid = 1$;*
- with binary conflict *iff $\mid X \mid > 2 \Rightarrow \emptyset \vdash X$;*
- with flow order *iff for all $e \in E$, it holds:*

  (a) $Con_1(e) \Rightarrow \downarrow^{rc} e \vdash \{e\}$, for all $\downarrow^{rc} e \subseteq E$,

  (b) $X \vdash_{min} \{e\} \wedge Con_2(X) \Rightarrow X = \downarrow^{rc} e$, for some $\downarrow^{rc} e \subseteq E$;

- in standard form *iff* $\vdash = \{(A, B) \mid A \cap B = \emptyset,\ A \cup B \in LC(\mathcal{E})\}$.

Example 2. *As an example, consider the RC-structure $\mathcal{E}^{rc} = (E^{rc}, \vdash^{rc}, L, l^{rc})$ from [11], where $E^{rc} = \{a, b, c\}$; $\vdash^{rc}$ consists of $\emptyset \vdash X$ for all $X \neq \{a, b\}$ and $\{c\} \vdash \{a, b\}$; $L = E^{rc}$; and l^{rc} is the identity labeling function. The RC-structure is rooted, pure, non-singular, with binary conflict and flow order. It is easy to see that $LC(\mathcal{E}^{rc}) = Conf(\mathcal{E}^{rc}) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a, c\},\ \{b, c\},\ \{c, a, b\}\}$. This RC-structure models the initial conflict between the events a and b that can be resolved by the occurrence of the event c. The structure $\mathcal{E}^{rc}$ can be presented in standard form $\widetilde{\mathcal{E}}^{rc}$ with $\widetilde{\vdash}^{rc}$ consisting of $A \widetilde{\vdash} B$ such that $B \subseteq C \in LC(\mathcal{E})$ and $A = C \setminus B$, i.e. $\widetilde{\vdash}^{rc} = \{(\emptyset, \emptyset),\ (\emptyset, \{a\}),\ (\{a\}, \emptyset),\ (\emptyset, \{b\}),\ (\{b\}, \emptyset),\ (\emptyset, \{c\}),\ (\{c\}, \emptyset),\ (\emptyset, \{a, c\}),\ (\{a, c\}, \emptyset),\ (\{a\}, \{c\}),\ (\{c\}, \{a\}),\ \ldots,\ (\emptyset, \{a, b, c\}),\ (\{a, b, c\}, \emptyset)\}$.*

Lemma 2.

(i) *Any RC-structure* $\mathcal{E} = (E, \vdash, L, l)$ *can be transformed into the RC-structure* $SF(\mathcal{E}) = (E, \widetilde{\vdash}, L, l)$[2] *in standard form such that* $LC(\mathcal{E}) = LC(SF(\mathcal{E}))$*. Moreover,* $SF(\mathcal{E})$ *is rooted if* $\mathcal{E}$ *is rooted;*

(ii) $Conf(\mathcal{E}) = Conf(SF(\mathcal{E}))$*, if* $\mathcal{E}$ *is a pure RC-structure.*

An event is called *impossible* (non-executable) if it does not occur in any of the configurations. In RC-structures, events can be impossible because of their unspecified enabling relations, or because of their infinite causes or impossible causes/predecessors.

Standard form of an RC-structure and the ability to specify impossible events in the model allows for developing a relatively simple structural definition of a removal operator.

Definition 5. *For an RC-structure* $\mathcal{E} = (E, \vdash, L, l)$ *in standard form and* $X \in LC(\mathcal{E})$*, a removal operator is defined as follows:* $\mathcal{E} \setminus X = (E', \vdash', L, l')$*, where*

$$
\begin{array}{lcl}
E' & = & E \setminus X \\
\vdash' & = & \{(A', B') \mid \exists (A, B) \in \vdash \ s.t. \ A' = A \cap E', \quad B' = B \cap E', \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (A' \cup B' \cup X) \in LC(\mathcal{E})\} \\
l' & = & l\,|_{E'}
\end{array}
$$

According to the definition above, all the events in X are removed; however, we retain the events, not forming left-closed sets with the events in X and hence conflicting with some events in X, making the retained events impossible by deleting their enabling relations.

Consider properties of the removal operator.

Lemma 3. *Given an RC-structure* $\mathcal{E}$ *in standard form and* $X \in Conf(\mathcal{E})$*,*

(i) $X \subseteq Y \in LC(\mathcal{E}) \iff Y \setminus X \in LC(\mathcal{E} \setminus X)$*.*

(ii) $\mathcal{E} \setminus X$ *is a rooted RC-structure in standard form;*

(iii) $X \subseteq Y \in LC(\mathcal{E})$ *and* $Y' \in LC(\mathcal{E})$ *for all* $X \subseteq Y' \subseteq Y \Rightarrow Y \setminus X \in Conf(\mathcal{E} \setminus X)$*.*

[2] An RC-structure $SF(\mathcal{E}) = (E, \widetilde{\vdash}, L, l)$ can be directly obtained by putting $\widetilde{\vdash} = \{(A, B) \mid B \subseteq C \in LC(\mathcal{E}), A = C \setminus B\}$.

We translate F-structures into RC-structures and conversely. Given an F-structure $F = (E, \sharp, \prec, L, l)$ over L and $e \in E$, define $\downarrow e$ as a maximal subset of E such that $\forall e' \in \downarrow e\colon e' \prec e$ and $\forall e' \neq e'' \in \downarrow e\colon \neg(e' \sharp e'')$. For an F-structure $F = (E, \sharp, \prec, L, l)$, let $\mathcal{RC}(F) = (E' = E, \vdash', L, l = l')$, where

$$X \vdash' Y \iff \begin{cases} \text{either } Y = \{e\}, \neg(e \sharp e), X = \downarrow e, \\ \text{or } Y = \{e, e'\}, e \neq e', \neg(e \sharp e'), X = \emptyset, \\ \text{or } \mid Y \mid \neq 1, 2, X = \emptyset. \end{cases}$$

For an RC-structure $RC = (E, \vdash, L, l)$, let $\mathcal{F}(RC) = (E, \sharp^{rc}, \prec^{rc}, L, l)$.

Lemma 4.

(i) For F an F-structure, $\mathcal{RC}(F)$ is a rooted, pure, singular RC-structure with binary conflict and flow order s.t. $Conf(F) = Conf(\mathcal{RC}(F))$.

(ii) For RC a rooted, pure, singular RC-structure with binary conflict and flow order, $\mathcal{F}(RC)$ is an F-structure s.t. $Conf(RC) = Conf(\mathcal{F}(RC))$.

2.3. Different semantics

In this subsection, we define interleaving and step semantics for the event structure models under consideration. From now on, we treat $\mathcal{E}$ as an event structure over L specified in Definitions 1 and 3, if not defined otherwise.

We first introduce auxiliary notations. Given an event structure $\mathcal{E}$ over L and configurations $X, X' \in Conf(\mathcal{E})$, we write

- $X \rightarrow_{int} X'$ iff $X \subseteq X'$ and $\mid X' \setminus X \mid = 1$;
- $X \rightarrow_{step} X'$ iff $X \subseteq X'$ and $X'' \in Conf(\mathcal{E})$, for all $X \subseteq X'' \subseteq X'$.

For an event structure $\mathcal{E}$ over L and $\star \in \{int, step\}$, a configuration $X \in Conf(\mathcal{E})$ is a *configuration in $\star$-semantics* of $\mathcal{E}$ iff $\emptyset \rightarrow_\star^* X$, where $\rightarrow_\star^*$ is the reflexive and transitive closure of $\rightarrow_\star$. Let $Conf_\star(\mathcal{E})$ denote the set of configurations in $\star$-semantics of $\mathcal{E}$.

Lemma 5. *Given an event structure $\mathcal{E}$ over L and $\star \in \{int, step\}$, $Conf(\mathcal{E}) = Conf_\star(\mathcal{E})$.*

Proposition 1. *Let $\mathcal{E}$ be an F-structure or a rooted RC-structure in standard form, and $\star \in \{int, step\}$. Then,*

(i) for any $\mathcal{E}' = \mathcal{E} \setminus X$, with $X \in Conf(\mathcal{E})$, and $\mathcal{E}'' = \mathcal{E}' \setminus X'$, with $X' \in Conf(\mathcal{E}')$, $X \cup X' \in Conf(\mathcal{E})$ and $\mathcal{E}'' = \mathcal{E} \setminus (X \cup X')$;

(ii) for any $X, X'' \in Conf(\mathcal{E})$ such that $X \rightarrow_\star X''$, $X'' \setminus X \in Conf(\mathcal{E} \setminus X)$ and, moreover, $\emptyset \rightarrow_\star X'' \setminus X$ in $\mathcal{E} \setminus X$.

For $\star \in \{int, step\}$, an event structure $\mathcal{E}$ over L, and configurations $X, X' \in Conf(\mathcal{E})$ such that $X \to_\star X'$, we write

- $l_{int}(X' \setminus X) = a$ iff $X' \setminus X = \{e\}$ and $l(e) = a$, if $\star = int$;
- $l_{step}(X' \setminus X) = M$ iff $M(a) = |\{e \in X' \setminus X \mid l(e) = a\}|$, for all $a \in L$, if $\star = step$.

3. Transition systems $TC(\mathcal{E})$ and $TR(\mathcal{E})$

In this section, we first give some basic definitions concerning labeled transition systems, and then define the mappings $TC(\mathcal{E})$ and $TR(\mathcal{E})$, which associate two distinct kinds of transition systems – one whose states are configurations and the other whose states are residual event structures – with an event structure $\mathcal{E}$ over L.

A transition system $T = (S, \to, i)$ over a set $\mathcal{L}$ of labels consists of a set of states S, a transition relation $\to \subseteq S \times \mathcal{L} \times S$, and an initial state $i \in S$. Two transition systems over $\mathcal{L}$ are *isomorphic* if their states can be mapped one-to-one to each other, preserving transitions and initial states. We call a relation $R \subseteq S \times S'$ a *bisimulation* between transition systems T and T' over $\mathcal{L}$ iff $(i, i') \in R$, and for all $(s, s') \in R$ and $l \in \mathcal{L}$: if $(s, l, s_1) \in \to$, then $(s', l, s'_1) \in \to$ and $(s_1, s'_1) \in R$, for some $s'_1 \in S'$; and if $(s', l, s'_1) \in \to$, then $(s, l, s_1) \in \to$ and $(s_1, s'_1) \in R$, for some $s_1 \in S$.

We need an additional auxiliary notation. For a fixed set L of labels of event structures, define $\mathbb{L}_{int} := L$, $\mathbb{L}_{step} := \mathbb{N}_0^L$ (the set of multisets over L, or functions from L to the non-negative integers), being other sets of labels of the transition systems.

We are ready to define labeled transition systems with configurations as states.

Definition 6. *For an event structure $\mathcal{E}$ over L, and $\star \in \{int, step\}$, $TC_\star(\mathcal{E})$ is the transition system $(Conf(\mathcal{E}), \rightharpoonup_\star, \emptyset)$ over $\mathbb{L}_\star$, where $X \xrightharpoonup{p}_\star X'$ iff $X \to_\star X'$ and $p = l_\star(X' \setminus X)$ in $\mathcal{E}$.*

For an event structure $\mathcal{E}$ over L and $\star \in \{int, step\}$, define $Reach_\star(\mathcal{E}) = \{\mathcal{F} \mid \exists \mathcal{E}_0, \ldots, \mathcal{E}_k \ (k \geq 0) \text{ s.t. } \mathcal{E}_0 = \mathcal{E},\ \mathcal{E}_k = \mathcal{F}, \text{ and } \mathcal{E}_i \rightharpoonup_\star^X \mathcal{E}_{i+1}\ (i < k)\}$, where $\mathcal{E}_i \rightharpoonup_\star^X \mathcal{E}_{i+1}$ iff $\exists X \in Conf(\mathcal{E}_i)$: $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X$ and $\emptyset \to_\star X$ in $\mathcal{E}_i$;

Lemma 6. *Given an event structure $\mathcal{E}$ over L and $\star \in \{int, step\}$, $Reach_{int}(\mathcal{E}) = Reach_{step}(\mathcal{E}) = Reach(\mathcal{E})$.*

Consider the definition of labeled transition systems with residuals as states.

Definition 7. *For an event structure $\mathcal{E}$ over L and $\star \in \{int, step\}$, $TR_\star(\mathcal{E})$ is the transition system $(Reach(\mathcal{E}),\ \rightharpoonup_\star,\ \mathcal{E})$ over $\mathbb{L}_\star$, where $\mathcal{F} \overset{p}{\rightharpoonup}_\star \mathcal{F}'$ iff $\mathcal{F} \rightharpoonup^X_\star \mathcal{F}'$, for some $X \in Conf(\mathcal{F})$, and $p = l_\star(X)$ in $\mathcal{F}$.*

Proposition 2. *Let $\mathcal{E}$ be an F-structure or a rooted RC-structure in standard form, and $\star \in \{int,\ step\}$. Then,*

(i) for any $X \in Conf(\mathcal{E})$, $\mathcal{E} \setminus X \in Reach(\mathcal{E})$;

(ii) for any $\mathcal{E}' \in Reach(\mathcal{E})$, there exists $X \in Conf(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X$;

(iii) for any $X', X'' \in Conf(\mathcal{E})$, if $X' \xrightarrow{p}_\star X''$ in $TC_\star(\mathcal{E})$, then $\mathcal{E} \setminus X' \overset{p}{\rightharpoonup}_\star \mathcal{E} \setminus X''$ in $TR_\star(\mathcal{E})$;

(iv) for any $\mathcal{E}', \mathcal{E}'' \in Reach(\mathcal{E})$, if $\mathcal{E}' \overset{p}{\rightharpoonup}_\star \mathcal{E}''$ in $TR_\star(\mathcal{E})$, then there are $X', X'' \in Conf(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X'$, $\mathcal{E}'' = \mathcal{E} \setminus X''$, and $X' \xrightarrow{p}_\star X''$ in $TC_\star(\mathcal{E})$.

Theorem 1. *Let $\mathcal{E}$ be an F-structure or a rooted RC-structure in standard form, and $\star \in \{int,\ step\}$. Then, $TC_\star(\mathcal{E})$ and $TR_\star(\mathcal{E})$ are isomorphic.*

Theorem 2. *For $\star \in \{int, step\}$, it holds:*

(i) $TR_\star(\mathcal{E})$ and $TR_\star(SF(\mathcal{RC}(\mathcal{E})))$ are isomorphic, if $\mathcal{E}$ is an F-structure;

(ii) $TR_\star(SF(\mathcal{E}))$ and $TR_\star(\mathcal{F}(\mathcal{E}))$ are isomorphic, if $\mathcal{E}$ is a rooted, pure, singular RC-structure with binary conflict and flow order.

4. Concluding remarks

In this paper, we treated two event-oriented models of concurrent processes: flow event structures, being a generalization of prime event structures, and event structures for resolvable conflict, being the most expressive event-oriented model. In particular, we define removal operators for the models under consideration and demonstrated that the configuration-based and residual-based transition systems belonging to a single event structure are isomorphic, in interleaving and step semantics. Also, translations from flow event structures into structures for resolvable conflict and back have been developed to exhibit expressiveness capabilities of the former in comparison with those of the latter. Finally, we have shown that our translations preserve residual-based transition systems up to isomorphism.

Work on extending our approach (e.g., to precursor [7], probabilistic [23], and local [13] event structures, to event structures with dynamic causality [1] and to labeled event structures with invisible actions) is presently under way and has yielded promising intermediate results. Another future line of research is to extend our results on comparing two kinds of transition systems to the non-pure case of resolvable conflict structures [10] and to the multiset transition relation.

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Appendix

Proof of Lemma 1. (i) Clearly, $\sharp'$ is a symmetric relation, because $\sharp$ and $\{(e,e) \mid e \in \sharp(X)\}$ are symmetric relations. Next, since $\prec$ is an irreflexive relation and $\prec' \subseteq \prec$, $\prec'$ is an irreflexive relation as well. So, $\mathcal{E} \setminus X$ is an F-structure.

(ii) Assume $X \subseteq Y \in Conf(\mathcal{E})$. Then, $Y \setminus X$ is a finite and conflict-free set in $\mathcal{E}$, and, moreover, $Y \setminus X \cap \sharp(X) = \emptyset$. So, $Y \setminus X$ is a conflict-free set of events in $\mathcal{E} \setminus X$. Check that $Y \setminus X$ is left-closed up to conflict. Take arbitrary $e, d \in E \setminus X$ such that $e \in Y \setminus X$, $d \prec' e$, and $d \notin Y \setminus X$. Clearly, $e, d \notin X$ and $d \notin Y$. Moreover, we have that $d \prec e$ and $\neg(d \ \sharp \ x \prec e)$, for all $x \in X$, by the definition of $\prec'$. Due to $Y \in Conf(\mathcal{E})$, we can find $f \in Y$ such that $d \ \sharp \ f \prec e$. Then, $f \in Y \setminus X$. Hence, $d \ \sharp' \ f$. As $X \subseteq Y \in Conf(\mathcal{E})$ and $f \in Y$, it holds that $\neg(f \ \sharp \ x)$, for all $x \in X$. Hence, $f \prec' e$. Since Y does not contain flow cycles in $\mathcal{E}$ and $\prec' \subseteq \prec$, $Y \setminus X$ does not contain flow cycles in $\mathcal{E} \setminus X$. Thus, $Y \setminus X \in Conf(\mathcal{E} \setminus X)$. $\square$

Proof of Lemma 2. For the transformation, we can directly put $\widetilde{\vdash} = \{(A, B) \mid B \subseteq C \in LC(\mathcal{E}), A = C \setminus B\}$.

(i) Suppose $X \in LC(\mathcal{E})$. For any $Y \subseteq X$, take $Z := X \setminus Y$. Then $Z \subseteq X$ and $Z \ \widetilde{\vdash} \ Y$. So, $X \in LC(SF(\mathcal{E}))$. Conversely, suppose $X \in LC(SF(\mathcal{E}))$.

Then, there is $Z \subseteq X$ such that $Z \,\widetilde{\vdash}\, X$. By the definition of $\widetilde{\vdash}$, $X = Z \cup X \in LC(\mathcal{E})$. Thus, $LC(\mathcal{E}) = LC(SF(\mathcal{E}))$. It is easy to see that $SF(\mathcal{E})$ is rooted if $\mathcal{E}$ is rooted.

(ii) Assume that $\mathcal{E}$ is a pure RC-structure. Take an arbitrary $X \in Conf(SF(\mathcal{E}))$. This means $X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \,\widetilde{\vdash}\, Y$. Let $X_i = \{e_1, \ldots, e_i\}$ $(0 \leq i \leq n)$. Clearly, $X_i \in LC(SF(\mathcal{E}))$, for all $i \leq n$. Hence, $X_i \in LC(\mathcal{E})$, due to item (i). Take arbitrary $i \leq n$ and $Y \subseteq X_i$. Consider two possible cases:

- $e_i \in Y$ Since $X_i \in LC(\mathcal{E})$, there is $Z \subseteq X_i$ such that $Z \vdash Y$. Due to the fact that $\mathcal{E}$ is a pure RC-structure, $Z \cap Y = \emptyset$. Hence, $e_i \notin Z$. So, $Z \subseteq X_{i-1}$.
- $e_i \notin Y$ This means that $Y \subseteq X_{i-1}$. Since $X_{i-1} \in LC(\mathcal{E})$, there is $Z \subseteq X_{i-1}$ such that $Z \vdash Y$.

Thus, $X \in Conf(\mathcal{E})$.

Take an arbitrary $X \in Conf(\mathcal{E})$. Notice that $SF(\mathcal{E})$ is a pure RC-structure. Applying reasonings analogous to those in the proof of the opposite direction, we obtain that $X \in Conf(SF(\mathcal{E}))$. □

Proof of Lemma 3. (i) ($\Rightarrow$) First, notice that $X \cap Y = \emptyset$. Suppose $(X \cup Y) \in LC(\mathcal{E})$. Then, for all $\widetilde{Y} \subseteq Y$, $(\widetilde{Y} \cup \widehat{Y}) \in LC(\mathcal{E})$, where $\widehat{Y} = X \cup Y \setminus \widetilde{Y}$. As $\mathcal{E}$ is in standard form, $\widehat{Y} \vdash \widetilde{Y}$, for all $\widetilde{Y} \subseteq Y$ and the corresponding $\widehat{Y}$. Obviously, $(\widetilde{Y} \cup (\widehat{Y}' = \widehat{Y} \setminus X) \cup X) \in LC(\mathcal{E})$ and $\widehat{Y}' \subseteq Y$. Due to the definition of $\vdash'$, for all $\widetilde{Y} \subseteq Y$, there exists $\widehat{Y}' \subseteq Y$ such that $\widehat{Y}' \vdash' \widetilde{Y}$. Thus, $Y \in LC(\mathcal{E} \setminus X)$.

($\Leftarrow$) Assume $Y \in LC(\mathcal{E} \setminus X)$. Then, for Y there is $\widehat{Y} \subseteq Y$ such that $\widehat{Y} \vdash' Y$. By the definition of $\vdash'$, this implies that $(X \cup \widehat{Y} \cup Y) = (X \cup Y) \in LC(\mathcal{E})$.

(ii) We now show that $\mathcal{E} \setminus X$ is in standard form.

($\Rightarrow$) Suppose $A' \vdash' B'$. Then, we can find $A \vdash B$ such that $A' = A \cap E'$, $B' = B \cap E'$ and $(A' \cup B' \cup X) \in LC(\mathcal{E})$, due to the definition of $\vdash'$. Since $\mathcal{E}$ is in standard form, it holds that $A \cap B = \emptyset$. This implies that $A' \cap B' = \emptyset$. Thanks to item (i), we get that $(A' \cup B') \in LC(\mathcal{E} \setminus X)$.

($\Leftarrow$) Assume $C' \in LC(\mathcal{E} \setminus X)$. Take $B' \subseteq C'$ and $A' = C' \setminus B'$. According to item (i), $(C' \cup X) = (A' \cup B' \cup X) \in LC(\mathcal{E})$. Moreover, since $(A' \cup X) \cap B' = \emptyset$, we get that $A' \cup X \vdash B'$, due to $\mathcal{E}$ being in the standard form. Hence, $A' \vdash' B'$, by the definition of $\vdash'$.

(iii) Assume that $X \subseteq Y \in LC(\mathcal{E})$ and $Y' \in LC(\mathcal{E})$, for all $X \subseteq Y' \subseteq Y$. We shall show that $Y \setminus X \in Conf(\mathcal{E} \setminus X)$. Since $Y \in LC(\mathcal{E})$, we have that $Y \setminus X$ is finite. W.l.o.g. assume $Y \setminus X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$. Take an arbitrary $i \leq n$ and arbitrary set $A \subseteq \{e_1, \ldots, e_i\}$. Clearly, $X \subseteq X \cup A \subseteq Y$.

So, $X \cup A \in LC(\mathcal{E})$. Due to item (i), it holds that $A \in LC(\mathcal{E} \setminus X)$. By item (ii), we get that $\mathcal{E} \setminus X$ is in the standard form. Hence, $\emptyset \vdash_{\mathcal{E}\setminus X} A$ and $\emptyset \subseteq \{e_1, \dots, e_{i-1}\}$. Thus, $Y \setminus X \in \mathit{Conf}(\mathcal{E} \setminus X)$. □

Proof of Lemma 4. (i) By the definition of $\mathcal{RC}(F)$, it is clear that $\mathcal{RC}(F) = (E', \vdash', L, l)$ is a rooted, pure, singular RC-structure with binary conflict. It is routine to show that for all $e \in E$ such that $Con_1(\{e\})$ it holds that $A = \downarrow e \iff A = \downarrow^{rc} e$, for all $A \subseteq E$. Then, $\mathcal{RC}(F)$ has flow order, due to the definition of $\vdash'$. Next, check that $\mathit{Conf}(F) = \mathit{Conf}(\mathcal{RC}(F))$.

Assume $X \in \mathit{Conf}(F)$. Then, using Proposition 2.3 from [5], we get that $X = \{e_1, \dots, e_k\}$ $(k \geq 0)$ such that $\neg(e_i \sharp e_j)$, for all $i, j \leq k$, and for all $i \leq k$ and for all $e \in E$ it holds that if $e \prec e_i$ then there is $j < i$ such that $e = e_j$ or there is $j < i$ such that $e \sharp e_j \prec e_i$. Then, $\downarrow e_i \subseteq \{e_1, \dots, e_{i-1}\}$, for some $\downarrow e_i$ and for all $i \leq k$. Verify that $X \in \mathit{Conf}(\mathcal{RC}(F))$. Take arbitrary $1 \leq l \leq k$ and $A \subseteq \{e_1, \dots, e_l\}$. According to the definition of $\vdash'$, three cases are admissible. If $A = \{e_j\}$ for some $j \leq l$, then $\{e_1, \dots, e_{j-1}\} \supseteq \downarrow e_j \vdash' \{e_j\}$, for some $\downarrow e_j$, because $\neg(e_j \sharp e_j)$. If $\mid A \mid = 2$, i.e. $A = \{e_p, e_q\}$ and $e_p \neq e_q$, then $\emptyset \vdash' A$, thanks to $\neg(e_p \sharp e_q)$. If $\mid A \mid \neq 1, 2$, then $\emptyset \vdash' A$. Thus, $X \in \mathit{Conf}(\mathcal{RC}(F))$.

Next, suppose $X \in \mathit{Conf}(\mathcal{RC}(F))$, i.e. $X = \{e_1, \dots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and all $A \subseteq \{e_1, \dots, e_i\}$, there is $B \subseteq \{e_1, \dots, e_{i-1}\}$ such that $B \vdash' A$. Check that X is conflict-free. Take arbitrary $i, j \leq n$. Since $\{e_i, e_j\} \subseteq X$, we can find $B \subseteq X$ such that $B \vdash' \{e_i, e_j\}$. By the definition of $\vdash'$, we have that $\neg(e_i \sharp e_j)$. Take arbitrary $i \leq n$ and $e \in E$ such that $e \prec e_i$. As $\{e_i\} \subseteq X$, there is $B \subseteq \{e_1, \dots, e_{i-1}\}$ such that $B \vdash' \{e_i\}$. Due to the definition of $\vdash'$, we have $B = \downarrow e_i$, for some $\downarrow e_i$, thanks to $\neg(e_i \sharp e_i)$. Then, $\exists j < i$ such that $e = e_j$ or $\exists j < i$ such that $e \sharp e_j \prec e_i$. Hence, using Proposition 2.3 from [5], we get that $X \in \mathit{Conf}(F)$.

(ii) Assume that $RC = (E, \vdash, L, l)$ is a rooted, pure, singular RC-structure with binary conflict and flow order. By the definition of $\mathcal{F}(RC) = (E, \sharp^{rc}, \prec^{rc}, L, l)$, $\sharp^{rc}$ is a symmetric relation. Since RC is pure, $\prec^{rc}$ is an irreflexive relation. Hence, $\mathcal{F}(RC)$ is a flow event structure. Check that $\mathit{Conf}(RC) = \mathit{Conf}(\mathcal{F}(RC))$.

Suppose $X \in \mathit{Conf}(RC)$, i.e. $X = \{e_1, \dots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and all $A \subseteq \{e_1, \dots, e_i\}$, there is $B \subseteq \{e_1, \dots, e_{i-1}\}$ such that $B \vdash A$. Take arbitrary $i, j \leq n$. Since $\{e_i, e_j\} \subseteq X$, we can find $B \subseteq X$ such that $B \vdash \{e_i, e_j\}$. This implies that $\neg(e_i \sharp^{rc} e_j)$, for all $i, j \leq n$. Take arbitrary $i \leq n$ and $e \in E$ such that $e \prec^{rc} e_i$. As $\{e_i\} \subseteq X$, there is $B \subseteq \{e_1, \dots, e_{i-1}\}$ such that $B \vdash \{e_i\}$. Clearly, we can find $B' \subseteq B$ such that $B' \vdash_{min} \{e_i\}$. Since RC has flow order, we may conclude that $\downarrow^{rc} e_i = B'$, for some $\downarrow e_i$, because $Con_2(X)$, i.e. $Con_2(B')$. Then, $\exists j < i$ such that $e = e_j$ or $\exists j < i$ such that $e \sharp^{rc} e_j \prec^{rc} e_i$. Hence, using Proposition 2.3 from [5], we get that $X \in \mathit{Conf}(\mathcal{F}(RC))$.

Next, assume $X \in \mathit{Conf}(\mathcal{F}(RC))$. Then, using Proposition 2.3 from [5], we get that $X = \{e_1, \ldots, e_k\}$ $(k \geq 0)$ such that $\neg(e_i \sharp^{rc} e_j)$, for all $i, j \leq k$, and for all $i \leq k$ and for all $e \in E$ it holds that if $e \prec^{rc} e_i$ then there is $j < i$ such that $e = e_j$ or there is $j < i$ such that $e \sharp^{rc} e_j \prec e_i$. Clearly, we have that $Con_1(X)$, $Con_2(X)$, and, moreover, $\downarrow^{rc} e_i \subseteq \{e_1, \ldots, e_{i-1}\}$, for some $\downarrow^{rc} e_i$. We have to show that $X \in \mathit{Conf}(RC)$. Take arbitrary $1 \leq l \leq k$ and $A \subseteq \{e_1, \ldots, e_l\}$. Four possible cases are admissible. If $\mid A \mid= 0$ then $\{e_1, \ldots, e_{l-1}\} \supseteq \emptyset \vdash A$, as RC is rooted. Consider the case when $\mid A \mid= 1$, i.e. $A = \{e_i\}$ $(i \leq l)$. As $Con_1(\{e_i\})$ and $\downarrow^{rc} e_i \subseteq \{e_1, \ldots, e_{i-1}\}$, for some $\downarrow^{rc} e_i$, we get that $\{e_1, \ldots, e_{l-1}\} \supseteq \downarrow^{rc} e_i \vdash \{e_i\}$, thanks to RC having flow order. If $\mid A \mid= 2$, i.e. $A = \{e_p, e_q\}$ and $e_p \neq e_q$, then there is $B \subseteq E$ such that $B \vdash \{e_p, e_q\}$, because $Con_2(\{e_p, e_q\})$. Since $\mathcal{RC}$ is singular, we may conclude that $B = \emptyset \subseteq \{e_1, \ldots, e_{l-1}\}$. If $\mid A \mid\geq 3$ then $\{e_1, \ldots, e_{l-1}\} \supseteq \emptyset \vdash A$, as $\mathcal{RC}$ has binary conflict. Thus, $X \in \mathit{Conf}(RC)$. $\square$

Proof of Lemma 5. Consider the more complex case when $\mathcal{E}$ is an RC-structure.

Take an arbitrary $X \in \mathit{Conf}_{int}(\mathcal{E})$. This means that $\emptyset = X_0 \rightarrow_{int} X_1 \ldots X_{n-1} \rightarrow_{int} X_n = X$, where $n \geq 0$, $X_i \in \mathit{Conf}(\mathcal{E})$, $X_{i-1} \subseteq X_i$, and $X_i \setminus X_{i-1} = \{e_i\}$, for all $1 \leq i \leq n$. Clearly, $A \in \mathit{Conf}(\mathcal{E})$, for all $X_{i-1} \subseteq A \subseteq X_i$, $(1 \leq i \leq n)$. Then, we get that $\emptyset = X_0 \rightarrow_{step} X_1 \ldots X_{n-1} \rightarrow_{step} X_n = X$, i.e. $X \in \mathit{Conf}_{step}(\mathcal{E})$. So, $\mathit{Conf}_{int}(\mathcal{E}) \subseteq \mathit{Conf}_{step}(\mathcal{E})$.

By definition, $Conf_{step}(\mathcal{E}) \subseteq Conf(\mathcal{E})$.

Take an arbitrary $X \in \mathit{Conf}(\mathcal{E})$. Since $X \in \mathit{Conf}(\mathcal{E})$, we have $X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and all $A \subseteq \{e_1, \ldots, e_i\}$, there is $B \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $B \vdash A$. Clearly, $X_i = \{e_1, \ldots, e_i\} \in \mathit{Conf}(\mathcal{E})$, for all $i \leq n$. This means that $\emptyset = X_0 \rightarrow_{int} X_1 \rightarrow_{int} \ldots \rightarrow_{int} X_n = X$. Hence, $X \in \mathit{Conf}_{int}(\mathcal{E})$. So, $\mathit{Conf}(\mathcal{E}) \subseteq Conf_{int}(\mathcal{E})$.

Summing up all the inclusions above, we get that $\mathit{Conf}(\mathcal{E}) = \mathit{Conf}_{int}(\mathcal{E}) = \mathit{Conf}_{step}(\mathcal{E})$. $\square$

Proof of Proposition 1. Consider the proof of the more complex case when $\mathcal{E}$ is a rooted RC-structure in standard form.

(i) Assume that $X \in \mathit{Conf}(\mathcal{E})$ and $X' \in \mathit{Conf}(\mathcal{E}')$, where $\mathcal{E}' = \mathcal{E} \setminus X$. Due to Lemma 3(ii), $\mathcal{E} \setminus X$ is a rooted RC-structure in standard form. Since $X \in \mathit{Conf}(\mathcal{E})$, we get that $X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $A \subseteq \{e_1, \ldots, e_i\}$, there is $B \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $B \vdash_{\mathcal{E}} A$. As $X' \in \mathit{Conf}(\mathcal{E}')$, it holds that $X' = \{e'_1, \ldots, e'_m\}$ $(m \geq 0)$ such that for all $j \leq m$ and for all $A' \subseteq \{e'_1, \ldots, e'_j\}$, there is $B' \subseteq \{e'_1, \ldots, e'_{j-1}\}$ such that $B' \vdash_{\mathcal{E}'} A'$. Check that $X \cup X' = \{e_1, \ldots, e_n, e'_1, \ldots, e'_m\} = \{e_1, \ldots, e_{n+m}\} \in \mathit{Conf}(\mathcal{E})$. Take an arbitrary $i \leq n + m$. If $i \leq n$, then the result follows from the fact that $X \in \mathit{Conf}(\mathcal{E})$. Consider the case when $n < i \leq n + m$. Take an arbitrary set $A \subseteq \{e_1, \ldots, e_i\}$. Define $A' = A \cap E' = A \setminus X \subseteq \{e'_1, \ldots, e'_{i-n}\}$. Then, we can find $B' \subseteq \{e'_1, \ldots, e'_{i-n-1}\}$ such that $B' \vdash_{\mathcal{E}'} A'$.

By the definition of $\vdash_{\mathcal{E}'}$, $A' \cup B' \cup X \in LC(\mathcal{E})$. Since $A' = A \setminus X$, we have $A \subseteq A' \cup X \subseteq A' \cup B' \cup X$ and $(A' \cup B' \cup X) \setminus A \subseteq B' \cup X \subseteq \{e_1, \ldots, e_{i-1}\}$. Due to $\mathcal{E}$ being in standard form, $(A' \cup B' \cup X) \setminus A \vdash_{\mathcal{E}} A$. So, $X \cup X' \in Conf(\mathcal{E})$. Thus, $X \cup X' \in Conf(\mathcal{E})$, by Lemma 5.

Define $\mathcal{E}'' = (\mathcal{E} \setminus X) \setminus X'$ and $\widetilde{\mathcal{E}} = \mathcal{E} \setminus (X \cup X')$. We need to show that $\mathcal{E}'' = \widetilde{\mathcal{E}}$. Due to Lemma 3(ii), $\widetilde{\mathcal{E}}$ and $\mathcal{E}''$ are rooted RC-structures in standard form. Notice that $X \cup X' \in Conf(\mathcal{E})$. By definition, $E'' = E' \setminus X' = E \setminus X \setminus X' = E \setminus (X \cup X') = \widetilde{E}$, and $l'' = l \mid_{E''} = l \mid_{\widetilde{E}} = \widetilde{l}$. Verify that $LC(\mathcal{E}'') = LC(\widetilde{\mathcal{E}})$. Take an arbitrary $Y \in LC(\mathcal{E}'')$. According to Lemma 3(i), $Y \in LC(\mathcal{E}'') \iff Y \cup X' \in LC(\mathcal{E} \setminus X) \iff Y \cup X' \cup X \in LC(\mathcal{E}) \iff Y \in LC(\mathcal{E} \setminus (X \cup X')) = LC(\widetilde{\mathcal{E}})$. Then, $\vdash'' = \widetilde{\vdash}$.

(ii) Assume that $X, X'' \in Conf(\mathcal{E})$ and $X \rightarrow_\star X''$ in $\mathcal{E}$. Then, $X, X'' \in Conf(\mathcal{E})$ and $X \subseteq X''$, by definitions. Notice that $\mathcal{E} \setminus X$ is rooted and in standard form, due to Lemma 3(ii). Since $X \rightarrow_\star X''$ in $\mathcal{E}$, $X \subseteq X''$ and we have the following.

$\star = int$ Then, $\mid X' \mid = 1$. This implies $A \in Conf(\mathcal{E}) \subseteq LC(\mathcal{E})$, for all $X \subseteq A \subseteq X''$. Hence, $B \in Conf(\mathcal{E} \setminus X)$, for all $\emptyset \subseteq B = A \setminus X \subseteq X'$, by Lemma 3(iii). So, $\emptyset \rightarrow_{int} X'$ in $\mathcal{E} \setminus X$.

$\star = step$ Then, $A \in Conf(\mathcal{E}) \subseteq LC(\mathcal{E})$, for all $X \subseteq A \subseteq X''$. Hence, $B \in Conf(\mathcal{E} \setminus X)$, for all $\emptyset \subseteq B = A \setminus X \subseteq X'$, by Lemma 3(iii). So, $\emptyset \rightarrow_{step} X'$ in $\mathcal{E} \setminus X$. □

Proof of Lemma 6. Consider the more complex case when $\mathcal{E}$ is an RC-structure.

Take an arbitrary $\mathcal{F} \in Reach_{int}(\mathcal{E})$. Then, there are $\mathcal{E}_0, \ldots, \mathcal{E}_k$ $(k \geq 0)$ such that $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{E}_k = \mathcal{F}$, and $\mathcal{E}_i \rightharpoonup^{X_i}_{int} \mathcal{E}_{i+1}$, for all $i < k$, where $\mathcal{E}_i \rightharpoonup^{X_i}_{int} \mathcal{E}_{i+1}$ iff $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X_i$, for some $X_i \in Conf(\mathcal{E}_i)$ such that $\emptyset \rightarrow_{int} X_i$. This means that $\mid X_i \mid = 1$. Hence, $A \in Conf(\mathcal{E}_i)$, for all $\emptyset \subseteq A \subseteq X_i$ and for all $i < k$. This implies that $\emptyset \rightarrow_{step} X_i$, for all $i < k$. So, $\mathcal{E}_i \rightharpoonup^{X_i}_{step} \mathcal{E}_{i+1}$, for all $i < k$. Thus, $\mathcal{F} = \mathcal{E}_k \in Reach_{step}(\mathcal{E})$.

Take an arbitrary $\mathcal{F} \in Reach_{step}(\mathcal{E})$. Then, there are $\mathcal{E}_0, \ldots, \mathcal{E}_k$ $(k \geq 0)$ such that $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{E}_k = \mathcal{F}$, and $\mathcal{E}_i \rightharpoonup^{X_i}_{step} \mathcal{E}_{i+1}$, for all $i < k$, where $\mathcal{E}_i \rightharpoonup^{X_i}_{step} \mathcal{E}_{i+1}$ iff $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X_i$, for some $X_i \in Conf(\mathcal{E}_i)$ such that $\emptyset \rightarrow_{step} X_i$. This means that $A \in Conf(\mathcal{E}_i)$, for all $\emptyset \subseteq A \subseteq X_i$ and for all $i < k$. Take an arbitrary $i < k$. W.l.o.g. assume that $X_i = \{e^i_1, \ldots, e^i_{n_i}\}$ $(n_i \geq 0)$. If $n_i = 0$ then $\mathcal{E}_i = \mathcal{E}_{i+1}$, by the definition of the removal operator, and the result is obvious. Consider the case with $n_i > 0$. Then, we get that $\{e^i_1\} \in Conf(\mathcal{E}_i)$ and, moreover, $\emptyset \rightarrow_{int} \{e^i_1\}$ in $\mathcal{E}_i$. Furthermore, $A' \in LC(\mathcal{E}_i)$, for all $\{e^i_1\} \subseteq A' \subseteq X_i$. By Lemma 3(iii), we have that $A'' \in Conf(\mathcal{E}_i \setminus \{e^i_1\})$, for all $\emptyset \subseteq A'' \subseteq X_i \setminus \{e^i_1\}$. Repeating the reasonings for all $1 < j \leq n_i$, we get that $\{e^i_j\} \in Conf(\mathcal{E}_i \setminus \{e^i_1\} \setminus \ldots \setminus \{e^i_{j-1}\})$ and, moreover, $\emptyset \rightarrow_{int} \{e^i_j\}$

in $\mathcal{E}_i \setminus \{e_1^i\} \setminus \ldots \setminus \{e_{j-1}^i\}$, for all $1 \leq j \leq n_i$. This implies that $\mathcal{E}_i \rightharpoonup_{int}^{e_1^i} \mathcal{E}_i \setminus \{e_1^i\} \ldots \mathcal{E}_i \setminus \{e_1^i\} \setminus \ldots \setminus \{e_{n_i-1}^i\} \rightharpoonup_{int}^{e_{n_i}^i} \mathcal{E}_i \setminus \{e_1^i\} \setminus \ldots \setminus \{e_{n_i}^i\}$. Thanks to Proposition 1(i), we get that $\mathcal{E}_i \setminus \{e_1^i\} \setminus \ldots \setminus \{e_{n_i}^i\} = \mathcal{E}_i \setminus \{e_1^i, \ldots, e_{n_i}^i\} = \mathcal{E}_{i+1}$. Due to the arbitrary choice of i, $\mathcal{F} = \mathcal{E}_k \in Reach_{int}(\mathcal{E})$. $\square$

Proof of Proposition 2. Consider the proof of the more complex case when $\mathcal{E}$ is a rooted RC-structure in standard form.

(i) Take an arbitrary set $X \in Conf(\mathcal{E})$. By Lemma 5, we get that $X_0 = \emptyset \rightarrow_\star X_1 \ldots X_{n-1} \rightarrow_\star X_n = X$ $(n \geq 0)$ in $\mathcal{E}$. This implies that $X_i \in Conf(\mathcal{E})$, for all $i < n$. Notice that $\mathcal{E} \setminus X_i$ $(i < n)$ is rooted and in standard form, due to Lemma 3(ii). We shall proceed by induction on n.

$n = 0$. $\mathcal{E} \setminus X_0 = \mathcal{E} \setminus \emptyset = \mathcal{E} \in Reach(\mathcal{E})$.

$n = 1$. By the induction hypothesis, $\mathcal{E} \setminus X_0 \in Reach(\mathcal{E})$. Since $X_0 \rightarrow_\star X_1$ in $\mathcal{E}$, it holds $X_1 \setminus X_0 \in Conf(\mathcal{E} \setminus X_0)$ and $\emptyset \rightarrow_\star X_1 \setminus X_0$ in $\mathcal{E} \setminus X_0$, by Proposition 1(ii). Clearly, $\mathcal{E} \setminus X_0 \setminus (X_1 \setminus X_0) = \mathcal{E} \setminus X_1$. This implies that $\mathcal{E} \setminus X_0 \rightharpoonup_\star^{X_1 \setminus X_0} \mathcal{E} \setminus X_1$. So, $\mathcal{E} \setminus X_1 \in Reach(\mathcal{E})$.

$n > 1$. By the induction hypothesis, $\mathcal{E} \setminus X_{n-2} \rightharpoonup_\star^{X_{n-1} \setminus X_{n-2}} \mathcal{E} \setminus X_{n-1} \in Reach(\mathcal{E})$. Since $X_{n-1} \rightarrow_\star X_n$ in $\mathcal{E}$, it holds $X_n \setminus X_{n-1} \in Conf(\mathcal{E} \setminus X_{n-1})$ and $\emptyset \rightarrow_\star X_n \setminus X_{n-1}$ in $\mathcal{E} \setminus X_{n-1}$, by Proposition 1(ii). Next, due to Proposition 1(i), we have $\mathcal{E} \setminus X_{n-1} \setminus (X_n \setminus X_{n-1}) = \mathcal{E} \setminus (X_{n-1} \cup (X_n \setminus X_{n-1})) = \mathcal{E} \setminus X_n$. This implies that $\mathcal{E} \setminus X_{n-1} \rightharpoonup_\star^{X_n \setminus X_{n-1}} \mathcal{E} \setminus X_n$. Thus, $\mathcal{E} \setminus (X_n = X) \in Reach(\mathcal{E})$.

(ii) Take an arbitrary $\mathcal{E}' \in Reach(\mathcal{E})$. Due to Lemma 6, we get that $\mathcal{E} = \mathcal{E}_0 \rightharpoonup_\star^{X_0} \mathcal{E}_1 \ldots \mathcal{E}_{n-1} \rightharpoonup_\star^{X_{n-1}} \mathcal{E}_n = \mathcal{E}'$ $(n \geq 0)$. By the definition of $\rightharpoonup_\star$, it holds that $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X_i$, for some $X_i \in Conf(\mathcal{E}_i)$ such that $\emptyset \rightarrow_\star X_i$ in $\mathcal{E}_i$ $(i < n)$. Notice that $\mathcal{E}_i$ $(i \leq n)$ is rooted and in standard form, due to Lemma 3(ii). Verify that $Y_i = \bigcup_{j=0}^{i} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_{i+1} = \mathcal{E} \setminus Y_i$, for all $i < n$. We shall proceed by induction on i.

$i = 0$. Then, $Y_0 = X_0 \in Conf(\mathcal{E} = \mathcal{E}_0)$ and $\mathcal{E}_1 = \mathcal{E}_0 \setminus X_0 = \mathcal{E} \setminus Y_0$.

$i > 0$. By the induction hypothesis, $Y_{i-1} = \bigcup_{j=0}^{i-1} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_i = \mathcal{E} \setminus Y_{i-1}$. Check that $Y_i = \bigcup_{j=0}^{i} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_{i+1} = \mathcal{E} \setminus Y_i$. As $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X_i$, it holds that $\mathcal{E}_{i+1} = (\mathcal{E} \setminus Y_{i-1}) \setminus X_i$. According to Proposition 1(i), we have that $Y_{i-1} \cup X_i \in Conf(\mathcal{E})$ and $\mathcal{E}_{i+1} = \mathcal{E} \setminus (Y_{i-1} \cup X_i) = \mathcal{E} \setminus Y_i$.

(iii) Take arbitrary $X', X'' \in Conf(\mathcal{E})$ such that $X' \xrightarrow{p}_\star X''$ in $TC_\star(\mathcal{E})$. Then, $X' \rightarrow_\star X''$, and $l_\star(X'' \setminus X') = p$ in $\mathcal{E}$. Moreover, by item (i), there are

$\mathcal{E} \setminus X'$, $\mathcal{E} \setminus X'' \in \mathit{Reach}(\mathcal{E})$. Notice that $\mathcal{E} \setminus X'$ and $\mathcal{E} \setminus X''$ are rooted and in standard form, due to Lemma 3(ii). We have to show that $\mathcal{E} \setminus X' \xrightharpoonup{p}_\star \mathcal{E} \setminus X''$ in $TR_\star(\mathcal{E})$. According to Proposition 1(ii), we get that $X'' \setminus X' \in \mathit{Conf}(\mathcal{E} \setminus X')$ and $\emptyset \rightarrow_\star X'' \setminus X'$ in $\mathcal{E} \setminus X'$. Then, $\mathcal{E} \setminus X' \rightharpoonup_\star^{X'' \setminus X'} \mathcal{E} \setminus X' \setminus (X'' \setminus X')$. According to Proposition 1(i), we get that $\mathcal{E} \setminus X' \setminus (X'' \setminus X') = \mathcal{E} \setminus X''$. Due to the definition of the removal operator, $l_\star(X'' \setminus X') = p$ in $\mathcal{E} \setminus X'$, if $\star \in \{\mathit{int}, \mathit{step}\}$. Hence, $\mathcal{E} \setminus X' \xrightharpoonup{p}_\star \mathcal{E} \setminus X''$ in $TR_\star(\mathcal{E})$, for $\star \in \{\mathit{int}, \mathit{step}\}$.

(iv) Take arbitrary $\mathcal{E}', \mathcal{E}'' \in \mathit{Reach}(\mathcal{E})$ such that $\mathcal{E}' \xrightharpoonup{p}_\star \mathcal{E}''$ in $TR_\star(\mathcal{E})$. By Lemma 3(ii), $\mathcal{E}'$ and $\mathcal{E}''$ are rooted and in standard form. We have to show that there are $X', X'' \in \mathit{Conf}(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X'$, $\mathcal{E}'' = \mathcal{E} \setminus X''$, and $X' \xrightarrow{p}_\star X''$ in $TC_\star(\mathcal{E})$. Due to item (ii), there is $X' \in \mathit{Conf}(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X' \in \mathit{Reach}(\mathcal{E})$. According to the definition of $\xrightharpoonup{p}_\star$ in $TR_\star(\mathcal{E})$, there is $\widetilde{X}' \in \mathit{Conf}_\star(\mathcal{E}')$ such that $\mathcal{E}'' = \mathcal{E}' \setminus \widetilde{X}'$, $\emptyset \rightarrow_\star \widetilde{X}'$ and $l_\star(\widetilde{X}') = p$ in $\mathcal{E}'$. Then, $X'' = X' \cup \widetilde{X}' \in \mathit{Conf}(\mathcal{E})$ and $\mathcal{E}'' = \mathcal{E} \setminus X''$, by Proposition 1(i). Consider two possible cases.

$\star = \mathit{int}$ Since $\emptyset \rightarrow_{\mathit{int}} \widetilde{X}'$ in $\mathcal{E}'$, we have $\widetilde{X}' = \{e\}$. Hence, $X' \subseteq X''$ and $| X'' \setminus X' = \widetilde{X}' | = 1$. So, $X' \rightarrow_{\mathit{int}} X''$ in $\mathcal{E}$. Moreover, $l_{\mathit{int}}(X'' \setminus X') = p$ in $\mathcal{E}$, by the definition of the removal operator. Thus, $X' \xrightarrow{p}_{\mathit{int}} X''$ in $\mathcal{E}$.

$\star = \mathit{step}$ Due to $\emptyset \rightarrow_{\mathit{step}} \widetilde{X}'$ in $\mathcal{E}'$, we get that $A \in \mathit{Conf}(\mathcal{E}')$, for all $\emptyset \subseteq A \subseteq \widetilde{X}'$. Then, by Proposition 1(i), $X''' = X' \cup A \in \mathit{Conf}(\mathcal{E})$, for all $\emptyset \subseteq A \subseteq \widetilde{X}'$. Hence, $X' \rightarrow_{\mathit{step}} X''$ in $\mathcal{E}$. Moreover, $l_{\mathit{step}}(X'' \setminus X') = p$ in $\mathcal{E}$, by the definition of the removal operator. Thus, $X' \xrightarrow{p}_{\mathit{step}} X''$ in $\mathcal{E}$. □

Proof of Theorem 1. Consider the proof of the more complex case when $\mathcal{E}$ is a rooted RC-structure in standard form.

Define a mapping $g : \mathit{Conf}(\mathcal{E}) \rightarrow \mathit{Reach}(\mathcal{E})$ as follows: $g(X) = \mathcal{E} \setminus X$, for all $X \in \mathit{Conf}(\mathcal{E})$. Clearly, $g(\emptyset) = \mathcal{E}$. Due to Proposition 2(i), $g(X)$ is well-defined.

Check that g is a bijective mapping. Suppose that $g(X) = g(X')$, for some $X, X' \in \mathit{Conf}(\mathcal{E})$. This means that $\mathcal{E} \setminus X = \mathcal{E} \setminus X'$. By the definition of the removal operator, we get that $E \setminus X = E \setminus X'$. Since $X, X' \subseteq E$, we have that $X = X'$. Thus, g is an injective mapping. Take an arbitrary $\mathcal{E}' \in \mathit{Reach}(\mathcal{E})$. Due to Proposition 2(ii), we get that $\mathcal{E}' = \mathcal{E} \setminus X$, for some $X \in \mathit{Conf}(\mathcal{E})$. So, g is a surjective mapping.

According to Propositions 2(iii) and 2(iv) and the fact that g is a bijective mapping, we have that $X \xrightarrow{p}_\star X'$ in $TC_\star(\mathcal{E})$ iff $g(X) \xrightharpoonup{p}_\star g(X')$ in $TR_\star(\mathcal{E})$. Thus, g is indeed an isomorphism. □

Proof of Theorem 2. (i) Assume that $\mathcal{E}$ is an F-structure. According to Lemma 4(i), it holds that $\mathcal{RC}(\mathcal{E})$ is a rooted, pure, singular RC-structure with binary conflict and flow order such that $Conf(\mathcal{E}) = Conf(\mathcal{RC}(\mathcal{E}))$. Furthermore, by Lemmas 2(i) and 2(ii), we get that $SF(\mathcal{RC}(\mathcal{E}))$ is a rooted RC-structure in standard form, and $Conf(\mathcal{RC}(\mathcal{E})) = Conf(SF(\mathcal{RC}(\mathcal{E})))$. Therefore, $Conf(\mathcal{E}) = Conf(SF(\mathcal{RC}(\mathcal{E})))$. From this fact, it follows that $X \rightarrow_\star X'$ in $\mathcal{E}$, for all $X, X' \in Conf(\mathcal{E})$, iff $X \rightarrow_\star X'$ in $SF(\mathcal{RC}(\mathcal{E}))$, for all $X, X' \in Conf(SF(\mathcal{RC}(\mathcal{E})))$, by the definition of $\rightarrow_\star$ ($\star \in \{int, step\}$). Due to the definition of $\mathcal{RC}(\mathcal{E})$ and the construction of $SF(\mathcal{RC}(\mathcal{E}))$, we have that $l_{\mathcal{E}}(e) = l_{\mathcal{RC}(\mathcal{E})}(e) = l_{SF(\mathcal{RC}(\mathcal{E}))}(e)$, for all $e \in X' \setminus X$. Then, using Definition 6, it is easy to see that for all $X, X' \in Conf(\mathcal{E})$, it holds that $X \xrightarrow{p}_\star X'$ in $TC_\star(\mathcal{E})$ iff $X \rightarrow_\star X'$ and $p = l_\star(X' \setminus X)$ in $\mathcal{E}$, and for all $X, X' \in Conf(SF(\mathcal{RC}(\mathcal{E})))$, it holds that $X \rightarrow_\star X'$ and $p = l_\star(X' \setminus X)$ in $SF(\mathcal{RC}(\mathcal{E}))$ iff $X \xrightarrow{p}_\star X'$ in $TC_\star(SF(\mathcal{RC}(\mathcal{E})))$. This implies that $TC_\star(\mathcal{E})$ and $TC_\star(SF(\mathcal{RC}(\mathcal{E})))$ coincide. Thanks to Theorem 1, $TC_\star(\mathcal{E})$ and $TR_\star(\mathcal{E})$ are isomorphic, because $\mathcal{E}$ is an F-structure. Since $SF(\mathcal{RC}(\mathcal{E}))$ is a rooted RC-structure in standard form, we get that $TC_\star(SF(\mathcal{RC}(\mathcal{E})))$ and $TR_\star(SF(\mathcal{RC}(\mathcal{E})))$ are isomorphic, again thanks to Theorem 1. Hence, $TR_\star(\mathcal{E})$ and $TR_\star(SF(\mathcal{RC}(\mathcal{E})))$ are isomorphic.

(ii) Suppose that $\mathcal{E}$ is a rooted, pure, singular RC-structure with binary conflict and flow order. Due to Lemma 4(ii), it holds that $\mathcal{F}(\mathcal{E})$ is an F-structure such that $Conf(\mathcal{E}) = Conf(\mathcal{F}(\mathcal{E}))$. Moreover, by Lemmas 2(i) and 2(ii), we get that $SF(\mathcal{E})$ is a rooted RC-structure in standard form, and $Conf(\mathcal{E}) = Conf(SF(\mathcal{E}))$. Therefore, $Conf(\mathcal{F}(\mathcal{E})) = Conf(SF(\mathcal{E}))$. From this fact, it follows that $X \rightarrow_\star X'$ in $\mathcal{F}(\mathcal{E})$, for all $X, X' \in Conf(\mathcal{F}(\mathcal{E}))$, iff $X \rightarrow_\star X'$ in $SF(\mathcal{E})$, for all $X, X' \in Conf(SF(\mathcal{E}))$, by the definition of $\rightarrow_\star$ ($\star \in \{int, step\}$). Due to the construction of $SF(\mathcal{E})$ and the definition of $\mathcal{F}(\mathcal{E})$, we have that $l_{SF(\mathcal{E})}(e) = l_{\mathcal{E}}(e) = l_{\mathcal{F}(\mathcal{E})}(e)$, for all $e \in X' \setminus X$. Furthermore, using Definition 6, it is easy to see that for all $X, X' \in Conf(SF(\mathcal{E}))$, it holds that $X \xrightarrow{p}_\star X'$ in $TC_\star(SF(\mathcal{E}))$ iff $X \rightarrow_\star X'$ and $p = l_\star(X' \setminus X)$ in $SF(\mathcal{E})$, and for all $X, X' \in Conf(\mathcal{F}(\mathcal{E}))$, it holds that $X \rightarrow_\star X'$ and $p = l_\star(X' \setminus X)$ in $\mathcal{F}(\mathcal{E})$ iff $X \xrightarrow{p}_\star X'$ in $TC_\star(\mathcal{F}(\mathcal{E}))$. This implies that $TC_\star(SF(\mathcal{E}))$ and $TC_\star(\mathcal{F}(\mathcal{E}))$ coincide. Thanks to Theorem 1, $TC_\star(SF(\mathcal{E}))$ and $TR_\star(SF(\mathcal{E}))$ are isomorphic, because $SF(\mathcal{E})$ is a rooted RC-structure in standard form. Since $\mathcal{F}(\mathcal{E})$ is an F-structure, $TC_\star(\mathcal{F}(\mathcal{E}))$ and $TR_\star(\mathcal{F}(\mathcal{E}))$ are isomorphic, again thanks to Theorem 1. Hence, $TR_\star(SF(\mathcal{E}))$ and $TR_\star(\mathcal{F}(\mathcal{E}))$ are isomorphic.